Indian Statistical Institute, Bangalore Centre.
Back-paper Exam : Graph Theory

Instructor : Yogeshwaran D.

Date : June 6, 2022.

Max. points : 50. **Time Limit : 3 hours.**

Answer as many questions as you can.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention it clearly. See the end of the question paper for notations.

1. Let G be a finite, simple, undirected graph and $s \neq t \in V(G)$. State and prove Menger's theorem for edge disjoint paths from s to t in G. (**10**)

2. Let Q_n be the hypercube graph on $\{0,1\}^n$. What are $\alpha(Q_n), \alpha'(Q_n), \beta(Q_n), \beta'(Q_n)$?

3. Let $p_r(G)$ denote the number of partitions of $V(G)$ into r non-empty independent sets. Show that $P(G,x) = \sum_{r=1}^{n} p_r(G) x_{(r)}$ where $x_{(r)} = x(x-1)\ldots(x-r+1)$ and $n = |V(G)|$. (**10**)

4. Compute the eigenvalues of the Laplacian and Adjacency matrices of the complete graph K_n for all $n \geq 1$. (**10**)

5. Consider a finite simple directed graph $G = (V, E)$ and $s \neq t \in V$. Further assume that there are no incoming edges at s or no out-going edges at t. If f is an integral flow of strength k, show that there exist k directed paths $p_1, \ldots, p_k$ such that for all $e \in E$, $|\{p_i : e \in p_i\}| \leq f(e)$. (**10**)

Some notations :

- Unless mentioned, G is assumed to be a finite, undirected, simple graph everywhere.

- Q_n is the hypercube graph on $\{0,1\}^n$ i.e., $V = \{0,1\}^n$ and $x \sim y$ is x and y differ exactly in one-coordinate i.e., $|\{i : x_i \neq y_i\} = 1$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.
- $P(G, x)$ - chromatic polynomial of the graph G.
- $\alpha'(G)$ - Maximum independent edge set. $\beta'(G)$ - Minimum edge cover.
- $\alpha(G)$ - Maximum independent vertex set ; $\beta(G)$ - Minimum vertex cover.